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When does $SC(X) = \mathbb{R}^X$ hold?

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ABSTRACT

A map $f : X \to Y$ between topological spaces is called scatteredly continuous if for each non-empty subspace $A \subset X$ the restriction $f|_A$ has a point of continuity. By $SC(X)$ we denote the set of all scatteredly continuous maps from X to the space of real numbers $\mathbb{R}$. We consider the following problem: What conditions must satisfy space X so that $SC(X) = \mathbb{R}^X$?

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Recall that a map $f : X \to Y$ between topological spaces is called scatteredly continuous if for each non-empty subspace $A \subset X$ the restriction $f|_A$ has a point of continuity. By $SC(X)$ we denote the set of all scatteredly continuous maps from X to the space of real numbers $\mathbb{R}$.

Obviously, the space $C(X)$ of all continuous functions from X to $\mathbb{R}$ coincides with $\mathbb{R}^X$ if and only if X is a discrete space. It is also clear that, if X is scattered, then $SC(X)$ coincides with $\mathbb{R}^X$. We consider the following problem: What conditions must satisfy space X so that $SC(X) = \mathbb{R}^X$?

Hewitt [1] called a space resolvable if it contains two disjoint dense subsets and irresolvable otherwise. A space is strongly irresolvable if every open subspace is irresolvable. A space X is maximal (regular maximal) if the topology of the space X is maximal in the collection of all crowded (regular crowded) topologies on X. By a crowded space we mean a space that contains no isolated points. By $\mathbb{R}$ we denote the space of real numbers; ω stands for the space of finite ordinals (= non-negative integers) endowed with the discrete topology. The rest of the notation and terminology is standard and can be found in [2].

Proposition 1. *If $SC(X) = \mathbb{R}^X$ and $\emptyset \neq Y = \bigcup\{Y_i : i \in \omega\} \subseteq X$, then there is $i \in \omega$ such that $Int_Y Y_i \neq \emptyset$.*

Proof. Suppose that $SC(X) = \mathbb{R}^X$ and $Y = \bigcup\{Y_i : i \in \omega\}$ is a non-empty subspace of X.

Without loss of generality, we can assume that $Y_i \cap Y_j = \emptyset$ for $i \neq j$. Define a map $f : X \to \mathbb{R}$ as follows:

$$f(y) = \begin{cases} i, & \text{if } y \in Y_i \\ 0, & \text{otherwise.} \end{cases}$$

Since $f \in SC(X)$, there is a point of continuity $y \in Y$ of the restriction map $f|_Y$. Then there is $i_0 \in \omega$ such that $y \in Y_{i_0}$. Thus, $f(y) = i_0$. For the neighborhood $(i_0 - \frac{1}{2}, i_0 + \frac{1}{2})$ there is a neighborhood $O(y)$ such that $f(O(y)) \subset (i_0 - \frac{1}{2}, i_0 + \frac{1}{2})$. And from the construction of the map f, this means that $f(O(y)) = i_0$. Therefore $O(y) \subset Y_{i_0}$ and $Int_Y Y_{i_0} \neq \emptyset$. □

Corollary 1. *If $SC(X) = \mathbb{R}^X$, then every countable subspace of the space X is scattered.*

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In [3], Malykhin attempted to prove that every crowded space is a countable union of subsets with empty interiors provided that every set is constructible ($[V = L]$). This fact and the previous proposition yield the following.

Proposition 2. *Let $[V = L]$. Then $SC(X) = \mathbb{R}^X$ if and only if the space X is scattered.*

Proposition 3. *If there is a crowded regular space X which is not a union of countably many subsets with empty interiors, then there is a crowded completely regular 0-dimensional space Y such that $SC(Y) = \mathbb{R}^Y$.*

Proof. Suppose that the space (X, τ) is not a countable union of subsets with empty interiors. Let τ' be a maximal regular topology on X such that $\tau \subseteq \tau'$. Clearly, the space (X, τ') is not a union of countably many subsets each having empty interior. Due to Lemma 2 in [3], there is a non-empty open subspace Y of (X, τ') in which the family of sets with empty interiors is a σ-ideal. Obviously, the space Y is regular maximal, and this means (see [4]) that it is a completely regular 0-dimensional maximal space.

Let us show that $SC(Y) = \mathbb{R}^Y$. Let $f \in \mathbb{R}^Y$. For every $n \in \omega$ we fix a countable family $\mathcal{V}_n$ of subsets of $\mathbb{R}$ such that $\bigcup \mathcal{V}_n = \mathbb{R}$ and $\operatorname{diam} V < \frac{1}{n+1}$ for each $V \in \mathcal{V}_n$. It is easy to see that $Y = \bigcup\{f^{-1}(V);\ V \in \mathcal{V}_n\}$ for every $n \in \omega$. Since a family of sets with empty interiors is a σ-ideal in the space Y, for every non-empty open set U in Y and for each $n \in \omega$ there is $V \in \mathcal{V}_n$ such that $\operatorname{Int}(f^{-1}(V) \bigcap U) \neq \emptyset$. For every $n \in \omega$, fix the maximal disjoint family $\mathcal{U}_n$ of open sets in Y such that $\operatorname{diam} f(U) < \frac{1}{n+1}$ for all $U \in \mathcal{U}_n$. Put $\mathrm{U}_n = \bigcup \mathcal{U}_n$ for every $n \in \omega$. Clearly, $\overline{\mathrm{U}_n} = Y$ for all $n \in \omega$. Since the family of sets with empty interiors is a σ-ideal in the space Y, $\overline{\bigcap\{\mathrm{U}_n : n \in \omega\}} = Y$. It is easy to understand that map f is continuous at every point of the set $Z = \bigcap\{\mathrm{U}_n : n \in \omega\}$. Since the space Y is maximal and Z is dense in Y, the subspace $Y \setminus Z$ is scattered [4]. Thus, $f \in SC(Y)$. □

Theorem 1 ([5]). *If ZFC is consistent with the existence of a measurable cardinal, so is ZFC with the existence of a completely regular 0-dimensional SIB (strongly irresolvable Baire space).*

Remark 1. As Kunen informed us, assuming the existence of any SIB space, one can construct an SIB space that contains a countable subspace A which is closed, nowhere dense in X and homeomorphic to the rationals. Then $f|_A$ could be discontinuous everywhere.

Since every SIB space is a crowded space which is not a union of countably many subsets with empty interior, applying Proposition 3 and Theorem 1, we have the following.

Corollary 2. *If ZFC is consistent with the existence of a measurable cardinal, so is ZFC with the existence of a completely regular 0-dimensional crowded space X such that $SC(X) = \mathbb{R}^X$.*

Proposition 2 and Corollary 2 imply the following.

Theorem 2. *The statement "$SC(X) = \mathbb{R}^X$ if and only if X is scattered" does not depend on ZFC.*

Now let us consider some special cases, namely, at which conditions on the space X the equality $SC(X) = \mathbb{R}^X$ implies that X is scattered.

Proposition 4. *Let $\mathcal{R}$ be a topological property such that any crowded space with this property is resolvable. If $SC(X) = \mathbb{R}^X$ and there is an open scatteredly continuous injective map from the space X to a space Y, each closed subspace of which contains a dense subspace with property $\mathcal{R}$, then X is scattered.*

Proof. Let $f : X \to Y$ be an open scatteredly continuous injective map from the space X to a space Y which has property $\mathcal{R}$, and let $SC(X) = \mathbb{R}^X$. Let us prove that $SC(Y) = \mathbb{R}^Y$.

Assume that ψ is an arbitrary map from a space Y to the space of real numbers $\mathbb{R}$. Consider the composition $\varphi = \psi \circ f$. Obviously, $\varphi : X \to \mathbb{R}$ is scatteredly continuous. Now let $B \subset Y$, $B \neq \emptyset$ and $A = f^{-1}(B)$. It is known that $f|_A : A \to B$ is an open map. Since φ is scatteredly continuous, there is a point $x_0 \in A$, which is a point of continuity of the map $\varphi|_A : A \to \mathbb{R}$. Let us show that the map $\psi|_B : B \to \mathbb{R}$ is continuous at $y_0 = f(x_0)$.

Let $O(\psi(y_0))$ be a neighborhood of the point $\psi(y_0)$ in $\mathbb{R}$. Since $\varphi(x_0) = \psi(y_0)$, there is a neighborhood $O(x_0)$ of the point x_0 in the subspace A such that $\varphi(O(x_0)) \subset O(\psi(y_0))$. Since $f|_A : A \to B$ is an open map, $f(O(x_0))$ is a neighborhood of the point y_0. Clearly, $\psi(f(O(x_0))) = \varphi(O(x_0)) \subset O(\psi(y_0))$. Thus, ψ is scatteredly continuous and $SC(Y) = \mathbb{R}^Y$.

Since $SC(Y) = \mathbb{R}^Y$ and each close subspace of Y contains a dense subspace with property $\mathcal{R}$, Y is scattered. Let us show that X is scattered too.

Assume that A is an arbitrary subset of X. Let B be the set of continuity points of the map $f|_A$. Obviously, the set B is dense in A. Since $f|_B$ is a scatteredly continuous map to a scattered space, there is an isolated point x^* in B. Since B is dense in A, the point x^* is isolated in A too. Therefore, the space X is scattered. □

It is known that the crowded k-spaces (see [6,7]) and the crowded countably compact Tychonoff spaces (see [8]) are resolvable.

Corollary 3. *If $SC(X) = \mathbb{R}^X$, then the space X is scattered if it satisfies one of the following conditions:*

1. *each close subspace of X contains a dense k-space;*
2. *each close subspace of X contains a dense countably compact Tychonoff space;*
3. *space X is k-scattered (i.e. for an arbitrary close subset $F \subset X$ there is a non-empty open $U \subset X$ such that $\overline{U \cap F}$ is compact).*

The proof of (3) follows from the statement that in every close subspace of k-scattered space there is an open dense locally compact subspace.

Remark 2. It is clear that. if $SC(X) = \mathbb{R}^X$, then X is hereditarily irresolvable; that is, every subspace of the space X is irresolvable. On the other hand, there exist countable crowded hereditarily irresolvable spaces (for example, the maximal countable spaces). Obviously, for such spaces we have $SC(X) \neq \mathbb{R}^X$.

We recall that a topological space X has countable tightness if for every subset $A \subseteq X$ and any point $a \in \overline{A}$ there is a countable subset $B \subseteq A$ with $a \in \overline{B}$.

Proposition 5 ([9]). *A map $f : X \to Y$ defined on a space X of countable tightness is scatteredly continuous if and only if for every countable subspace $Q \subset X$ the restriction $f|_Q$ has a continuity point.*

Proof. The "only if" part of this proposition is trivial. To prove the "if" part, assume that $f : X \to Y$ is not scatteredly continuous and X has countable tightness. Since f is not scatteredly continuous, there is a subset $D \subset X$ such that $f|_D$ has no continuity point.

By induction on the tree $\omega^{<\omega}$ we can construct a sequence $(x_s)_{s\in\omega^{<\omega}}$ of points of D such that for every sequence $s \in \omega^{<\omega}$ the following conditions hold:

- x_s is a cluster point of the set $\{x_{s\wedge n} : n \in \omega\}$;
- $f(x_s)$ is not a cluster point of $\{f(x_{s\wedge n}) : n \in \omega\}$.

We start the inductive construction taking any point $x_\emptyset \in D$. Assuming that for some finite sequence $s \in \omega^{<\omega}$ the point $x_s \in D$ has been chosen, use the discontinuity of $f|_D$ at x_s to find a neighborhood U of $f(x_s)$ such that x_s is a cluster point of $D \setminus f^{-1}(U)$. Since X has countable tightness there is a countable set $\{x_{s\wedge n} : n \in \omega\} \subset D \setminus f^{-1}(U)$ whose closure contains the point x_s. This completes the inductive construction.

Then the set $Q = \{x_s : s \in \omega^{<\omega}\}$ is countable and $f|_Q$ has no continuity point. □

Theorem 3. *Let a space X have countable tightness. Then $SC(X) = \mathbb{R}^X$ if and only if X is scattered.*

Proof. Suppose that the tightness of space X is countable and $SC(X) = \mathbb{R}^X$. Due to Corollary 1, each countable subspace of X is scattered. Let τ_d be a discrete topology on the set X and $i : X \to (X, \tau_d)$ be the identity map. Clearly, the map $i|_Q : Q \to (X, \tau_d)$ is scatteredly continuous for arbitrary countable subspace $Q \subset X$. And by Proposition 5, this means that the map $i : X \to (X, \tau_d)$ is scatteredly continuous. Thus, the space X is scattered. □

Definition 1. A set A is called dividing if there is a non-empty set F such that $\overline{A \cap F} = \overline{F \setminus A}$, and A is called undividing if $\overline{A \cap F} \neq \overline{F \setminus A}$ for arbitrary non-empty set F.

Obviously, all close, open and scattered subsets of any space X are undividing.

Proposition 6. *The characteristic function χ_A of a set A of a space X is scatteredly continuous if and only if A is undividing.*

Proof. The "only if" part. Assume that the characteristic function of a set A is scatteredly continuous. Suppose that the set A is dividing. Then there is a subset F of X such that $\overline{A \cap F} = \overline{F \setminus A} = F$. Let x_0 be a point of continuity of the map $\chi_A|_F : F \to \mathbb{R}$. Then for arbitrary neighborhood Ox_0 we have

$$\emptyset \neq \chi_A(Ox_0 \cap A \cap F) \subseteq \chi_A(A) = \{1\}$$
$$\emptyset \neq \chi_A(Ox_0 \cap F \setminus A) \subseteq \chi_A(X \setminus A) = \{0\}.$$

This contradicts the fact that $\chi_A|_F$ is continuous at the point x_0.

The "if" part. Suppose that χ_A is not scatteredly continuous. Then there is a subset B of X such that the restriction χ_A has no continuity point on B. Let $P = B \cap A$ and $Q = B \cap (X \setminus A)$. Obviously, P and Q are non-empty sets. Clearly, $\overline{P} = \overline{Q}$. Otherwise, the map $\chi_A|_B$ is continuous at every point of the non-empty set $(P \setminus \overline{Q}) \cup (Q \setminus \overline{P}) \subset B$. This contradicts the fact that the restriction χ_A has no continuity points on B. Thus, $\overline{P} = \overline{Q}$ and $\overline{A \cap \overline{P}} = \overline{\overline{P} \setminus A} = \overline{P}$. □

By $\mathbb{D}$ we denote the two-point space $\{0, 1\}$ with discrete topology. And by $SC(X, \mathbb{D})$ we denote the set of all scatteredly continuous maps acting from X to $\mathbb{D}$.

Proposition 7. *The equation $SC(X, \mathbb{D}) = \mathbb{D}^X$ holds if and only if each subset of the space X is undividing.*

Recall that a space X is called perfectly paracompact if X is paracompact and each closed subset of X is of type G_δ in X. And a space X is paracompact if each open cover of X can be refined to a locally finite open cover.

Proposition 8. *If a space X is perfectly paracompact, then every undividing subset A of X is an F_σ-set and G_δ-set in X.*

Proof. Suppose there is a non-empty undividing not F_σ-set A in a perfectly paracompact space X. Denote by $\mathcal{U}$ the family of all open subsets U of X such that the intersection of each U with A is a non-empty F_σ-set in X.

Put $V = \cup\mathcal{U}$ (if $\mathcal{U} = \emptyset$; then $V = \emptyset$). Since V is open and X is perfectly normal there is a countable family $\mathcal{F} = \{F_i\}_{i\in\omega}$ of closed in X subsets such that $V = \bigcup \mathcal{F}$. Since V is an F_σ-set in X, V is a paracompact subspace. Let $\mathcal{V} = \{V_s\}_{s\in S}$ be a locally finite in V open refinement of $\mathcal{U}$. Since $V_s \cap A = V_s \cap (U \cap A)$, where $V_s \subseteq U \in \mathcal{U}$, $V_s \cap A$ is an F_σ-set for every $s \in S$.

Let $V_s \cap A = \bigcup\{\Phi_i^s : i \in \omega\}$, where Φ_i^s are closed subsets in X for all $i \in \omega$ and $s \in S$. Put $K_{ij} = F_i \cap (\bigcup\{\Phi_j^s : s \in S\})$ for all $i, j \in \omega$.

Local finiteness in V of the cover $\{V_s\}_{s\in S}$ of the set V implies that K_{ij} is a closed subset of F_i. Thus K_{ij} is a closed subset in X for all $i, j \in \omega$. It is easy to check that $V \cap A = \bigcup\{K_{ij} : i, j \in \omega\}$. Hence, $V \cap A$ is an open subset of A and an F_σ-set in X. Clearly, each open in A and F_σ-set in X is contained in $V \cap A$. Since A is not an F_σ-set, we have $A \setminus V \neq \emptyset$.

Let $F = \overline{A \setminus V}$. Obviously, $\overline{F \cap A} = F$. As A is an undividing set, then $\overline{F \setminus A} \neq F$. Then the set $W = A \setminus \overline{F \setminus A}$ is a non-empty open subset in A.

Let us show that $A \setminus \overline{F \setminus A} = (F \setminus \overline{F \setminus A}) \cup (A \cap V)$. Let $x \in A \setminus \overline{F \setminus A}$ and $x \notin A \cap V$. This means that $x \in A \setminus \overline{F \setminus A}$ and $x \in A \setminus V$. Then $x \in A \setminus \overline{F \setminus A}$ and $x \in F$, which means that $x \in F \setminus \overline{F \setminus A}$. Now let $x \in (F \setminus \overline{F \setminus A}) \cup (A \cap V)$. Two cases are possible: $x \in A \cap V$ or $x \in F \setminus \overline{F \setminus A}$. If $x \in A \cap V$, then $x \notin \overline{A \setminus V}$. And this means that $x \notin F$ and more so $x \notin \overline{F \setminus A}$. Then $x \in A$ and $x \notin \overline{F \setminus A}$. Hence, $x \in A \setminus \overline{F \setminus A}$. If $x \in F \setminus \overline{F \setminus A} \subset F \setminus (F \setminus A) \subset A$, this implies at once that $x \in A \setminus \overline{F \setminus A}$. Thus, $A \setminus \overline{F \setminus A} = (F \setminus \overline{F \setminus A}) \cup (A \cap V)$ and $W = A \setminus \overline{F \setminus A}$ is a union of two F_σ-sets in X. Therefore, W is an F_σ-set in X.

Clearly, W is not a subset of $V \cap A$. And this contradicts the fact that $V \cap A$ contains all open in A and F_σ-sets in X. Thus, all undividing sets of X are F_σ-sets in X. Since the complement of an undividing set is an undividing set, then all undividing sets of X are G_δ-sets in X too. $\square$

Recall that a space X is a Q-space if every subset of X is an F_σ-set.

Corollary 4. *If X is a perfectly paracompact space and $SC(X, \mathbb{D}) = \mathbb{D}^X$, then X is a Q-space.*

Recall that a space X is called Baire if the intersection of countably many dense open subsets of X is a dense subset of X. A space X is called a hereditary Baire space if each close subspace of X is a Baire space.

Proposition 9. *If a space X is hereditary Baire, then each F_σ-set and G_δ-set of X is undividing.*

Proof. Let A be an F_σ-set and G_δ-set in X. Suppose that A is dividing. Then there is a non-empty close subset F of X such that $\overline{A \cap F} = \overline{F \setminus A} = F$. Since the sets $A \cap F$ and $F \setminus A$ are dense G_δ-sets in the close subspace F, and X is a hereditary Baire space, $A \cap F \cap (F \setminus A) \neq \emptyset$. But $A \cap F \cap (F \setminus A) = \emptyset$, a contradiction. $\square$

From the preceding proposition and Proposition 8 we obtain the following theorem.

Theorem 4. *Let X be a hereditary Baire perfectly paracompact space. Then a subset A of X is undividing if and only if A is an F_σ-set and G_δ-set in X.*

Corollary 5. *Let X be a hereditary Baire perfectly paracompact space. Then $SC(X, \mathbb{D}) = \mathbb{D}^X$ if and only if X is a Q-space.*

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